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# Amplitude Phase-Space Model for Quantum Mechanics

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We show that there is a close relationship between quantum mechanics and ordinary probability theory. The main difference is that in quantum mechanics the probability is computed in terms of an amplitude function, while in probability theory a probability distribution is used. Applying this idea, we then construct an amplitude model for quantum mechanics on phase space. In this model, states are represented by amplitude functions and observables are represented by functions on phase space. If we now postulate a conjugation condition, the model provides the same predictions as conventional quantum mechanics. In particular, we obtain the usual quantum marginal probabilities, conditional probabilities and expectations. The commutation relations and uncertainty principle also follow. Moreover Schrödinger's equation is shown to be an averaged version of Hamilton's equation in classical mechanics.

## 1. AMPLITUDE FUNCTIONS

It is well known that probabilities are computed differently for quantum systems than for classical statistical systems (Feynman, 1948; Feynman and Hibbs, 1965; Gudder, 1984; Montroll, 1952). Roughly speaking, for classical systems the probability of an event is merely the sum of the probabilities of the outcomes composing the event. For a quantum system the probability is computed in terms of an amplitude function $\mathscr{A}$. The probability of a quantum event $A$ is found by summing $\mathscr{A}$ over the outcomes in $A$ and taking the modulus squared of this sum. The cross terms give interference effects which are characteristic of quantum systems but which are not present in classical ones.

Let us illustrate this by a simple example. Let $\Omega=\{\omega_1,\ldots,\omega_n\}$ be a finite set of outcomes corresponding to a statistical system $S$. If $S$ is classical,

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one has a probability function $P\colon \Omega \to [0, 1]$ satisfying $\Sigma P(\omega_i) = 1$. If $A \subseteq \Omega$ is an event, then the probability of $A$ is given by

$$P(A) = \Sigma_{\omega_i \in A} P(\omega_i)$$

Then $P$ has the usual properties of a probability measure, namely,

(a) $0 \leq P(A) \leq 1$

(b) $P(\Omega) = 1$

(c) $P(A \cup B) = P(A) + P(B) \qquad \text{if } A \cap B = \varnothing$

Notice that in this case any $\mathscr{A} \subseteq \Omega$ may be considered to be an event.

On the other hand, if $S$ is quantum mechanical, one has an amplitude function $\mathscr{A}\colon \Omega \to \mathbb{C}$ satisfying $|\Sigma \mathscr{A}(\omega_i)|^2 = 1$. If $A \subseteq \Omega$ is a quantum event then we define the quantum probability of $A$ by

$$P_q(A) = |\Sigma_{\omega_i \in A} \mathscr{A}(\omega_i)|^2$$

We immediately notice some very strange behaviors, which are sometimes called "quantum paradoxes." First of all we can have $P_q(A) > 1$ for some subsets $A \subseteq \Omega$. This shows that not all subsets of $\Omega$ can be considered to be quantum events. Also, even if $A$ and $B$ are quantum events and $A \cap B = \varnothing$ we may have $P_q(A \cup B) \neq P_q(A) + P_q(B)$. This may be interpreted as meaning that $A$ and $B$ are not simultaneously observable. That is, a measurement of $A$ may interfere with a measurement of $B$. Among the properties (a), (b), and (c) of a probability measure, the only one we can be sure of is property (b) that $P_q(\Omega) = 1$. As a simple example, let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and let $\mathscr{A}\colon \Omega \to \mathbb{C}$ be the amplitude function $\mathscr{A}(\omega_1) = \mathscr{A}(\omega_2) = 1$, $\mathscr{A}(\omega_3) = -1$. If $A = \{\omega_1, \omega_2\}$, then $P_q(A) = 4$ so $A$ is not a quantum event. If $B = \{\omega_1\}$, $C = \{\omega_3\}$, then $P_q(B) = P_q(C) = 1$ so $B$ and $C$ may be considered as quantum events. However, $B \cap C = \varnothing$ but $P_q(B \cup C) = 0 \neq P_q(B) + P_q(C)$.

Even though quantum systems exhibit some unusual behavior, they can still be studied in a systematic manner. Continuing our analogy in the previous paragraph, let $G\colon \Omega \to \mathbb{R}$; that is, $G$ is a random variable. If $S$ is classical, the expectation of $G$ is computed by $E(G) = \Sigma G(\omega_i) P(\omega_i)$. Again, $E$ has very nice properties:

(1) $E(1) = 1$

(2) $E(\alpha G_1 + \beta G_2) = \alpha E(G_1) + \beta E(G_2), \qquad \alpha, \beta \in \mathbb{R}$

(3) $E(G) \geq 0 \qquad \text{if } G \geq 0$

If $S$ is quantum mechanical, we must proceed differently. In this case only certain random variables can be interpreted as quantum observables. We can always write $G$ in the form $G = \Sigma \alpha_i \chi_{A_i}$ where $\alpha_i \neq \alpha_j$ for $i \neq j$. Now $G$ is a quantum observable only if the sets $A_i$ are quantum events. If this is the case, we define $E_q(G) = \Sigma \alpha_i P_q(A_i)$. Then conditions (1) and (3) hold for $E_q$ but (2) may fail.

In the sequel we shall primarily be interested in functions $G:\Omega\times\Omega\to\mathbb{R}$ of two variables. This case is far more interesting mathematically and more applicable for quantum mechanics. Assuming stochastic independence, the probability measure is given by the product measure $P((\omega_i, \omega_j)) = P(\omega_i)P(\omega_j)$. In the classical case, the expectation of $G$ is given by

$$E(G) = \Sigma_{i,j} P(\omega_i) G(\omega_i, \omega_j) P(\omega_j)$$

Then $E$ has the properties (1), (2), and (3) above. In the quantum case, the quantum expectation of $G$ is most naturally defined as

$$E_q(G) = \Sigma_{i,j} \mathscr{A}(\omega_i)^* G(\omega_i, \omega_j) \mathscr{A}(\omega_j)$$

where * denotes the complex conjugate. Now $E_q$ satisfies (1) and (2) above. However, $E_q$ may not satisfy (3). In general, $E_q(G)\geq 0$ only if the matrix $G_{ij} = G(\omega_i, \omega_j)$ is positive definite. Moreover, $E_q(G)$ need not even be real unless $G_{ij}$ is self-adjoint. This is essentially why self-adjoint operators are important in quantum mechanics.

Notice that in the classical case we have

$$E(G) = \Sigma_{i,j} P(\omega_i) G(\omega_i, \omega_j) P(\omega_j) = \Sigma_{i,j} P(\omega_i) G(\omega_j, \omega_i) P(\omega_j)$$

This property need not hold in the quantum case. To treat this systematically, we define the 1-amplitude of $G:\Omega\times\Omega\to\mathbb{R}$ as

$$\mathscr{A}_1(G)(\omega_i) = \Sigma_j G(\omega_i, \omega_j) \mathscr{A}(\omega_i)$$

Then the 1-expectation of $G$ is defined as

$$E_1(G) = \Sigma_i \mathscr{A}(\omega_i)^* \mathscr{A}_1(G)(\omega_i)$$

This is what we called $E_q(G)$ above. Similarly we define the 2-amplitude of $G$ as

$$\mathscr{A}_2(G)(\omega_j) = \Sigma_i G(\omega_i, \omega_j) \mathscr{A}(\omega_i)$$

and the 2-expectation as

$$E_2(G) = \Sigma_j \mathscr{A}(\omega_j)^* \mathscr{A}_2(G)(\omega_j)$$

In general, $E_1(G)\neq E_2(G)$ and in fact $E_1(G) = E_2(G)^*$. We have $E_1(G) = E_2(G)$ only if $G$ is self-adjoint.

This example motivates our work on an amplitude phase-space model for quantum mechanics in the next section, and the reader should keep it in mind. However, the theory in the next section will be more complicated for three basic reasons. First, we shall be working in a two-dimensional continuum $\mathbb{R}^2$ so sums will be replaced by integrals and amplitudes will be replaced by amplitude densities. Second, we will need some rather unusual measure theory on $\mathbb{R}^2$ (including some non-Lebesgue measurable sets).

Third, we shall not have stochastic independence and the amplitude densities will be functions of two variables.

## 2. PHASE-SPACE MODEL

There is a large body of literature on phase-space models for quantum mechanics (Cohen, 1966, 1976; Gudder, 1984; Pitowski, 1984; Srinivas and Wolf, 1975; Wigner, 1932). However, to our knowledge, this is the first which incorporates the concept of an amplitude density. This work was begun in Gudder (1984), but we now have more definitive results to report. There is a similarity between some of our results and those of Pitowski (1982, 1983, 1984). However, Pitowski does not use amplitude densities, and for this reason our work carries the theory further.

Let $\mathbb{R}^2=\{(q,p): q, p\in\mathbb{R}\}$ be a two-dimensional phase-space. For a function $f: \mathbb{R}^2\to\mathbb{C}$ define $f_q(\cdot)=f(q,\cdot)$ and $f_p(\cdot)=f(\cdot, p)$. We say that $f$ is *Q measurable* if $f_q$ is Lebesgue integrable for all $q\in\mathbb{R}$ and $g_1(q)=\int f(q,p)\,dp$ is Lebesgue measurable. Similarly, $f$ is *P measurable* if $f_p$ is Lebesgue integrable for all $p\in\mathbb{R}$ and $g_2(p)=\int f(q,p)\,dq$ is Lebesgue measurable. In general, a $Q$- or $P$-measurable function need not be Lebesgue measurable on $\mathbb{R}^2$ and most of the functions we consider will not be. A function $f: \mathbb{R}^2\to\mathbb{C}$ is an *amplitude density* if $f$ is both $Q$ measurable and $P$ measurable and

$$\int\left|\int f(q,p)\,dp\right|^2 dq=\int\left|\int f(q,p)\,dq\right|^2 dp=1$$

Notice, that this implies that $g_1\in L^2(\mathbb{R}, dq)$, $g_2\in L^2(\mathbb{R}, dp)$ and $\|g_1\|=\|g_2\|=1$. If $G:\mathbb{R}^2\to\mathbb{C}$ is Lebesgue measurable we write $G\in L^2(f)$ if

$$\int G(q,p)f(q,p)\,dp\in L^2(\mathbb{R}, dq)$$

and

$$\int G(q,p)f(q,p)\,dq\in L^2(\mathbb{R}, dp)$$

Our above remark shows that $1\in L^2(f)$.

We now make a series of definitions. For $G\in L^2(f)$ define the $Q(P)$ *amplitude* of $G$, respectively, by

$$\mathscr{A}^f_Q(G)(q)=\int G(q,p)f(q,p)\,dp$$

$$\mathscr{A}^f_P(G)(p)=\int G(q,p)f(q,p)\,dq$$

In the sequel we shall omit the $f$ and write $\mathscr{A}_Q(G)$ and $\mathscr{A}_P(G)$ when no confusion can result. For $A\in B(\mathbb{R})$ (Borel sets) define the amplitudes

$$\mathscr{A}_Q(A)(q)=\mathscr{A}_Q[\chi_A(p)](q)$$
$$\mathscr{A}_P(A)(p)=\mathscr{A}_P[\chi_A(q)](p)$$

and the *$Q(P)$-density reduced by an $A$ measurement of $P(Q)$*, respectively,

$$F(q,A)=|\mathscr{A}_Q(A)(q)|^2$$
$$F(A,p)=|\mathscr{A}_P(A)(p)|^2$$

For $A, B\in B(\mathbb{R})$ define the *$Q(P)$ probability of $B$ reduced by an $A$ measurement of $P(Q)$*, respectively,

$$\mu_Q(B,A)=\int_B F(q,A)\,dq$$

$$\mu_P(B,A)=\int_B F(A,p)\,dp$$

The *$Q(P)$ probability of $B$ following an $A$ measurement of $P(Q)$*, respectively, are defined as

$$\mu_Q(B|A)=\mu_Q(B,A)/\mu_Q(\mathbb{R},A)$$
$$\mu_P(B|A)=\mu_P(B,A)/\mu_P(\mathbb{R},A)$$

For $G\in L^2(f)$ we define the $Q(P)$ *expectation* of $G$, respectively, by

$$E_Q(G)=\int \mathscr{A}_Q^*(1)(q)\mathscr{A}_Q(G)(q)\,dq$$

$$E_P(G)=\int \mathscr{A}_P^*(1)(p)\mathscr{A}_P(G)(p)\,dp$$

Finally, if $E_Q(G)=E_P(G)$ we write $E(G)=E_Q(G)$ and call $E(G)$ the *expectation* of $G$.

The following lemma already shows the close similarity between the present theory and that of ordinary quantum mechanics:

*Lemma 1.* (a) $\mathscr{A}_Q(1)\in L^2(\mathbb{R},dq)$, $\mathscr{A}_P(1)\in L^2(\mathbb{R},dp)$, $\|\mathscr{A}_Q(1)\|=\|\mathscr{A}_P(1)\|=1$; (b) $E_Q[\chi_A(q)]=\mu_Q(A,\mathbb{R})=\int_A|\mathscr{A}_Q(1)|^2\,dq$, $E_P[\chi_A(p)]=\mu_P(A,\mathbb{R})=\int_A|\mathscr{A}_P(1)|^2\,dp$; (c) $\mathscr{A}_Q[G(q)]=G(q)\mathscr{A}_Q(1)$, $\mathscr{A}_P[G(p)]=G(p)\mathscr{A}_P(1)$; (d) $E_Q[G(q)]=\int G(q)|\mathscr{A}_Q(1)|^2\,dq$, $E_P[G(p)]=\int G(p)|\mathscr{A}_Q(1)|^2\,dp$; (e) $G\to\mathscr{A}_Q(G)$, $G\to\mathscr{A}_P(G)$, $G\to E_Q(G)$, $G\to E_P(G)$ are linear and $E_Q(1)=E_P(1)=1$.

*Proof.* Straightforward application of definitions. ■

For $\psi \in L^2(\mathbb{R}, dq)$, we denote the Fourier transform by

$$\hat{\psi}(p) = \frac{1}{(2\pi)^{1/2}} \int \psi(q)\, e^{-iqp/\hbar}\, dq$$

The inverse Fourier transform is denoted by $\check{\psi}(q)$. We say that an amplitude density $f$ is *regular* if $\mathscr{A}^f_Q(G)\hat{} = \mathscr{A}^f_P(G)$ for every $G = G(p)$ and $G = G(q)$. This is equivalent to the condition $\mathscr{A}^f_Q(A)\hat{} = \mathscr{A}^f_P(A)$ for all $A \in B(\mathbb{R})$. The requirement that $f$ is regular is called the *conjugation postulate.* The conjugation postulate imposes the condition that $\mathscr{A}^f_Q(G)$ and $\mathscr{A}^f_P(G)$ are Fourier conjugates of each other. Second, it introduces Planck's constant into the theory. The introduction of the constant $\hbar$ in the integral is justified on physical grounds since this makes $qp/\hbar$ dimensionless. The next theorem characterizes regular amplitude densities in terms of quantum mechanical states.

*Theorem 2.* An amplitude density $f$ is regular if and only if (1) for every $p \in \mathbb{R}$,

$$f(q, p) = (2\pi)^{-1/2} \mathscr{A}^f_Q(1)(q)\, e^{-iqp/\hbar} \text{ a.e. } [q]$$

(2) for every $q \in \mathbb{R}$

$$f(q, p) = (2\pi)^{-1/2} \mathscr{A}^f_Q(1)\hat{}(p)\, e^{iqp/\hbar} \text{a.e. } [p]$$

*Proof.* Let $f$ be an amplitude density that satisfies (1) and (2) and let $\psi = \mathscr{A}^f_Q(1)$. Then

$$\mathscr{A}^f_Q[G(p)] = \int G(p) f(q, p)\, dp$$

$$= \frac{1}{(2\pi)^{1/2}} \int G(p) \hat{\psi}(p)\, e^{iqp/\hbar}\, dp = (G\hat{\psi})\check{}(q)$$

Hence, $\mathscr{A}^f_Q[G(p)]\hat{} = G\hat{\psi}$. We then have

$$\mathscr{A}^f_P[G(p)] = \int G(p) f(q, p)\, dq$$

$$= G(p) \frac{1}{(2\pi)^{1/2}} \int \psi(q)\, e^{-iqp/\hbar}\, dq = G\hat{\psi} = \mathscr{A}^f_Q[G(p)]\hat{}$$

Moreover,

$$\mathscr{A}_P^f[G(q)] = \int G(q)f(q,p)\,dq$$

$$= \frac{1}{(2\pi)^{1/2}} \int G(q)\psi(q)\,e^{-iqp/\hbar}\,dq = (G\psi)\hat{}$$

and

$$\mathscr{A}_Q^f[G(q)] = \int G(q)f(q,p)\,dp = G\psi$$

Hence, $\mathscr{A}_Q^f[G(q)]\hat{} = \mathscr{A}_Q^f[G(q)]$ and $f$ is regular. Conversely, suppose $f$ is regular. Then

$$\int G(q)f(q,p)\,dq = \mathscr{A}_P^f[G(q)] = \mathscr{A}_Q^f[G(q)]\hat{}$$

$$= \frac{1}{(2\pi)^{1/2}} \int \mathscr{A}_Q^f[G(q)]\,e^{-iqp/\hbar}\,dq$$

$$= \frac{1}{(2\pi)^{1/2}} \int G(q)\mathscr{A}_Q^f(1)\,e^{-iqp/\hbar}\,dq$$

It follows that condition (1) holds. Condition (2) also holds since

$$\int G(p)f(q,p)\,dp = \mathscr{A}_Q^f[G(p)] = \mathscr{A}_P^f[G(p)]$$

$$= \frac{1}{(2\pi)^{1/2}} \int \mathscr{A}_P^f[G(p)]\,e^{iqp/\hbar}\,dp$$

$$= \frac{1}{(2\pi)^{1/2}} \int G(p)\mathscr{A}_P^f(1)\,e^{iqp/\hbar}\,dp \qquad \blacksquare$$

When $f$ is regular, we shall frequently denote $\mathscr{A}_Q^f(1)(q)$ simply by $\psi(q)$. Theorem 2 shows that if $f$ is regular, then it corresponds to a unique quantum state $\psi$ according to equations (1) and (2). Conversely, it is shown in Gudder (1984) that if $\psi$ is a quantum state, then there exists a regular $f$ corresponding to it according to equations (1) and (2). Lemma 1 now shows that for regular $f$ the usual quantum marginal probabilities and expectations are reproduced. In particular, $\mu_Q(A,\mathbb{R}) = \int_A |\psi(q)|^2\,dq$ and $\mu_P(A,\mathbb{R}) = \int_A |\hat{\psi}(p)|^2\,dp$. It was proved in Gudder (1984) that the conditional probabilities are the same as in traditional quantum mechanics. That is,

$$\mu_Q(B|A)=\mathrm{tr}[E^Q(B)E^P(A)P_\psi E^P(A)]/\mathrm{tr}[E^P(A)P_\psi]$$
$$\mu_P(B|A)=\mathrm{tr}[E^P(B)E^Q(A)P_\psi E^Q(A)]/\mathrm{tr}[E^Q(A)P_\psi]$$

where $P_\psi$ is the one-dimensional projection onto $\psi$ and $E^Q, E^P$ are the resolutions of identity for $Q$ and $P$, respectively.

We now assume that $\psi$ is a Schwartz test function, and define the usual quantum operators $(Q\psi)(q)=q\psi(q)$, $(P\psi)(q)=-\hbar i(\partial\psi/\partial q)(q)$, $(P\hat{\psi})(p)=p\hat{\psi}(p)$. The next theorem shows that the expectations of functions of $q$ and $p$ reproduce the usual quantum results.

*Theorem 3.* Let $f$ be a regular amplitude density with corresponding quantum state $\psi$. (a) If $G(q,p)=\Sigma a_{mn}q^m p^n$, then

$$\mathscr{A}_Q(G)=\Sigma a_{mn}Q^m P^n, \qquad \mathscr{A}_P(G)=\Sigma a_{mn}P^n(Q^m\psi)^\wedge$$
$$E_Q(G)=\langle\Sigma a_{mn}Q^m P^n\psi, \psi\rangle, \qquad E_P(G)=\langle\Sigma a_{mn}P^n Q^m\psi, \psi\rangle$$

(b) For any $G_1$, $G_2$ for which the expressions are defined

$$E_Q[G_1(q)+G_2(p)]=E_P[G_1(q)+G_2(p)]=\langle[G_1(Q)+G_2(P)]\psi, \psi\rangle$$

*Proof.* (a) We shall prove this result for a function of the form $G(q,p)=qp$. The proof for more general $G$ is similar:

$$\mathscr{A}_Q(qp)=\int qpf(q,p)\,dp=\frac{1}{(2\pi)^{1/2}}q\int p\hat{\psi}(p)\,e^{iqp/\hbar}\,dp$$
$$=q[-i\hbar(\partial\psi/\partial q)]=QP\psi$$
$$\mathscr{A}_P(qp)=\int qpf(q,p)\,dq=\frac{1}{(2\pi)^{1/2}}p\int q\psi(q)\,e^{-iqp/\hbar}\,dq$$
$$=p(q\psi)^\wedge=P(Q\psi)^\wedge$$

$$E_Q(qp)=\int \mathscr{A}_Q^*(1)(q)\mathscr{A}_Q(qp)(q)\,dq=\langle QP\psi, \psi\rangle$$

For the last equation, let $F$ denote the Fourier transform. Then

$$E_P(qp)=\int \mathscr{A}_P^*(1)(p)\mathscr{A}_P(qp)(p)\,dp=\langle p(Q\psi)^\wedge, \hat{\psi}\rangle$$
$$=\langle pF(Q\psi), F\psi\rangle=\langle F^*pFQ\psi, \psi\rangle=\langle PQ\psi, \psi\rangle$$

(b) The proof is similar to (a). ∎

The next corollary shows that the Heisenberg commutation relations and uncertainty principle hold in this theory.

*Corollary 4.* If $f$ is regular, then
(a) $E_Q(qp) - E_P(qp) = \langle [Q, P]\psi, \psi\rangle = i\hbar$
(b) $\Delta q \Delta p \geq \hbar/2$ where $\Delta q = [E(q^2) - E^2(q)]^{1/2}$ and $\Delta p = [E(p^2) - E^2(p)]^{1/2}$

We now consider Schrödinger's equation. Suppose the classical Hamiltonian is

$$H(q, p) = p^2/2m + V(q)$$

If the system is closed, then we have conservation of energy $H(q, p) = E$. Taking the $Q$ amplitude gives $\mathscr{A}_Q[H(q, p)] = \mathscr{A}_Q(E)$. Using the linearity of $\mathscr{A}_Q$, Lemma 1, and Theorem 3, we conclude that for a regular $f$

$$H(Q, P)\psi = E\psi$$

which is, of course, the time-independent Schrödinger equation. The classical dynamics is given by Hamilton's equation $dp/dt = -\partial H/\partial q$. Assuming this holds in the $Q$-amplitude average gives

$$\frac{d}{dt}\mathscr{A}_Q(p) = -\frac{\partial}{\partial q}\mathscr{A}_Q(H)$$

Hence,

$$\frac{\partial}{\partial q}\left[-i\hbar\frac{\partial\psi}{\partial t}\right] = -\frac{\partial}{\partial q}\mathscr{A}_Q(H)$$

Integrating both sides gives

$$i\hbar\frac{\partial\psi}{\partial t} = H(Q, P)\psi$$

which is the time-dependent Schrödinger equation. We conclude that Schrödinger's equation is an amplitude averaged version of Hamilton's equation for classical mechanics.

## 3. REALITY AND HIDDEN VARIABLES

This amplitude phase-space model can be thought of as a realistic, hidden-variables model for quantum mechanics. The hidden variables are the points $(q, p)$ of phase space themselves. If we know the point $(q, p)$, then we know the precise values of position and momentum. Thus, at least theoretically, the model is realistic since position and momentum have simultaneous precise values. In practice, however, all that we have available is an amplitude density $f(q, p)$. If $f$ is regular, then $f$ corresponds to a usual quantum state. A dynamical variable is now given by a function $G$ on $\mathbb{R}^2$ just as in classical mechanics. The corresponding quantum observable is obtained by averaging $G$ with the amplitude density. We have also seen that quantum mechanics is a stochastic version of classical mechanics. However, the averaging must be done correctly. It must be done in terms of an amplitude density and not as in traditional probability theory. What seems to be happening is that a particle moves classically in a hidden-variable phase space. If this motion is randomized by an amplitude density, then the wave function quantum dynamics is obtained.

Roughly speaking, we have proceeded as follows. The dynamical equation for nonrelativistic quantum mechanics is Schrödinger's equation $i(\partial\psi/\partial t) = H(Q, P)\psi$. The fact that $i$ appears in this equation is puzzling. Why should a physical dynamics be described by a complex-valued function? There is much literature on this question and many investigators have tried to answer it. Other researchers have tried to avoid the $i$ using various mathematical techniques. In our opinion neither of these attempts have been very successful. Our approach is to accept this $i$ and use it as our starting point for an axiomatic development of quantum mechanics. Taking $i$ together with the fact that quantum mechanics is probabilistic ($P$) leads us to amplitude densities on phase space. We can compute probabilities, conditional probabilities, and expectations using the amplitude densities. In the cases in which the conjugation postulate ($CP$) holds, we can recover quantum states. We then obtain the usual quantum expressions for the probabilities, conditional probabilities, and expectations. Taking the amplitude average of Hamilton's equation for classical mechanics ($CM$) brings us back to the quantum mechanical ($QM$) dynamics given by Schrödinger's equation. In short,

$$i + P + CP + CM = QM$$

## REFERENCES

Cohen, L. (1966). *J. Math. Phys.*, **7**, 781.
Cohen, L. (1976). *Found. Phys.*, **6**, 739.
Feynman, R. P. (1948). *Rev. Mod. Phys.*, **20**, 367.
Feynman, R. P., and Hibbs, A. R. (1965). *Quantum Mechanics and Path Integrals.* McGraw-Hill, New York.
Gudder, S. (1984). *J. Math. Phys.*, **25**, 456.
Gudder, S. (1984). Probability manifolds, *J. Math. Phys.*, **25**, 2397.
Montroll, E. W. (1952). *Commun. Pur. Appl. Math.*, **5**, 415.
Pitowski, I. (1982). *Phys. Rev. Lett.*, **48**, 1299.
Pitowski, I. (1983). *Phys. Rev.*, **D27**, 2316.

Pitowski, I. (1984). A phase-space model for quantum mechanics in which all operators commute, in *Fundamental Problems in Quantum Mechanics Conference Proceedings*, CUNY Albany, New York.

Srinivas, M., and Wolf, E. (1975). *Phys. Rev.*, **D11**, 1477.

Wigner, E. P. (1932). *Phys. Rev.*, **40**, 749.